Complex Dynamics Homework Solutions

1. Find the equilibrium of Beverton-Holt model.

   The Beverton-Holt model is a discrete-time model (shown here with annual generations):

$$S_{t+1} = S_t \frac{e^r}{1 + aS_t}. \tag{1}$$

   A discrete time model is at equilibrium then the left and right sides of the equation are identical, i.e.,

$$S^* = S^* \frac{e^r}{1 + aS^*}. \tag{2}$$

   Dividing both sides by S^* we have

$$1 = \frac{e^r}{1 + aS^*}, \tag{3}$$

   which can be rearranged to give the equilibrium value:

$$S^* = \frac{e^r - 1}{a}. \tag{4}$$

2. Consider a population with density dependent growth given by the Beverton-Holt model with parameters $r = 0.96$ and $a = 0.0044$, and initial population size $S = 147$. What will be the population size after five generations?

   This problem is solved by evaluating the equation 1 successively five times. In the first time step we have $S_{t+1} = 147\frac{e^{0.96}}{1+0.0044(147)} = 233.1305$. In the second generation we have $S_{t+1} = 233.1305\frac{e^{0.96}}{1+0.0044(233.1305)} = 300.5597$. Similarly, for the third, fourth, and fifth generations, we calculate 337.9907, 354.9146 and 361.8521.

3. What is the carrying capacity of the population in the previous question? Is it stable?

   From the solution to question 1, we calculate that the carrying capacity is $S^* = \frac{e^{0.96}-1}{0.0044} = 366.2947$. By evaluating equation 1 at values slightly smaller and slightly larger than this (say, 365 and 367) and observing that they both move in the direction of the equilibrium we conclude that the equilibrium is stable. Specifically, $365\frac{e^{0.96}}{1+0.0044(365)} = 365.7979$ and $367\frac{e^{0.96}}{1+0.0044(367)} = 366.5644$.

4. The version of the Beverton-Holt model introduced in this chapter does not consider harvesting. Modify equation 8 to represent the dynamics of a harvested fish population.

   We will assume that fish harvest occurs after recruitment and prior to reproduction. This means that of stock size S in a given year, only a portion $U < S$ is available for reproduction. The difference between U and S is the number of individuals harvested, H. That is

$$S - U = H. \tag{5}$$

   We will assume a harvest proportional to the stock size with catch rate h. Thus, $H = hS$. Substituting this last equation into 5 and rearranging, we have

$$U = S - hS. \tag{6}$$

   Now, we insert this into out basic Beverton-Holt model, yielding:

$$S_{t+1} = (S_t - hS_t)\frac{e^r}{1 + a(S_t - hS_t)}. \tag{7}$$

5. Another discrete time density-dependent model is the *logistic map*, $x_{t+1} = rx_t(1-x_t/k)$, which is named for its similarity to the continuous time logistic equation. Find the equilibria of this model. Iterate the model for a range of values of r. Plot the bifurcation diagram. How is the logistic map like the Ricker model? How is it different?

   To find the equilibria of the logistic map we set $x_{t+1} = x_t = x^*$ and solve for x^*, i.e.,

$$\begin{aligned}
x^* &= rx^*(1 - x^*/k) && \text{Divide by } x_t \\
1 &= r(1 - x^*/k) && \text{Multiply through on the right hand side} \\
1 &= r - rx^*/k && \text{Subtract } r \text{ and divide by } -1 \\
r - 1 &= rx^*/k && \text{Multiply by } k/r \\
k(r-1)/r &= x^*.
\end{aligned}$$

6. Sometimes, for instance to prepare data for time series analysis, it is useful to think about the dynamics of the logarithm of population size. Consider the change of variable $x = \ln(S)$. What is the difference equation for the logarithm of population size according to the Ricker model, *i.e.*, find the expression for f in the difference equation $x_{t+1} = f(x_t)$. Recall that on the ordinary scale and with annual generations rather than biannual generations, Ricker model dynamics are give by $S_{t+1} = S_t e^{r-bS_t}$.

   First we rearrange the identity $x = \ln(S)$ to five $S = e^x$. Then we substitute e^x in the Ricker model yielding

$$e^{x_{t+1}} = e^{x_t} e^{r-be^{x_t}} = e^{x_t + r - be^{x_t}}. \tag{8}$$

Since we are looking for a description of the dynamics of x we log-transform both sides to get

$$x_{t+1} = x_t + r - be^{x_t}. \tag{9}$$