Extinction Homework Solutions

1. Derive the coefficient of variation in equation 10 from equations 8 and 9.

   The coefficient of variation of a series of numbers is the ratio of its standard deviation σ to its mean m. The standard deviation is just the square root of the variance, so $\sigma(t) = \sqrt{v(t)}$. Combining equations 8 and 9, we have

$$CV = \frac{\sqrt{v(t)}}{m(t)} \tag{1}$$

   Expanding $v(t)$ and $m(t)$ yields

$$CV = \frac{\sqrt{n_0 \frac{b+d}{b-d} e^{(b-d)t} (e^{(b-d)t} - 1)}}{n_0 e^{(b-d)t}} \tag{2}$$

   Since $e^{(b-d)t}$ is large, $e^{(b-d)t} - 1 \approx e^{(b-d)t}$, giving

$$CV \approx \frac{\sqrt{n_0 \frac{b+d}{b-d} e^{(b-d)t} (e^{(b-d)t})}}{n_0 e^{(b-d)t}} = \frac{\sqrt{n_0 \frac{b+d}{b-d} \left[e^{(b-d)t}\right]^2}}{n_0 e^{(b-d)t}} \tag{3}$$

   By distribution (rule: $\sqrt{ab}\sqrt{a}\sqrt{b}$), we can rewrite this equation as

$$CV \approx \frac{\sqrt{n_0}\sqrt{\frac{b+d}{b-d}} e^{(b-d)t}}{n_0 e^{(b-d)t}}. \tag{4}$$

   We cancel the term $e^{(b-d)t}$ appearing in both the numerator and denominator, yielding

$$CV \approx \frac{\sqrt{n_0}\sqrt{\frac{b+d}{b-d}}}{n_0}. \tag{5}$$

   Now, since $\sqrt{a}/a = a^{1/2} - a^1 = a^{-1/2}$ we can combine the terms in n_0, yielding

$$CV \approx \sqrt{\frac{b+d}{b-d}} n_0^{-1/2} \tag{6}$$

Year (t)	N_t
1959	44
1960	47
1961	46
1962	44
1963	46
1964	45
1965	46
1966	40
1967	39
1968	39
1969	42
1970	39
1971	41
1972	40
1973	33

Finally, we recognize that the parameter combination $b - d$ is the same as our concept of the *intrinsic rate of increase*, designated by r. Making this substitution, we have the solution:

$$CV \approx \sqrt{\frac{b+d}{r} n_0}^{-1/2} \tag{7}$$

2. Estimate the expected time to extinction of the Yellowstone grizzly bear population from 1973, had the park not acted as it did.

   The trajectory of the data suggests that if the park had not acted as it did, then the population would have continued to decline. This decline is shown by the data from 1959 to 1973.

   Page 13 the reading gives a formula for the mean time to extinction,

   $$m(t) = \ln(n_0)/|r - v_2/2|, \tag{8}$$

   given initial population size n_0, intrinsic rate of increase r, and environmental variance v_2. Page 15 shows how to estimate r and v_2 from a data series and states that when applied to the portion of this time series from 1959 to 1973 obtains the estimates $\hat{r} = -0.018$ and $\hat{v}_2 = 0.006$. Inserting these into the equation and using $n_0 = 44$ (the population size in 1944) yields expected years to extinction of $m(t) = \ln(44)/| - 0.018 - 0.006/2| \approx 180$.

3. The example above assumes that the dynamics of Yellowstone grizzly bears should be considered in two epochs, before and after the 1973 legislation. However, it is possible that the legislation had no effect at all and that the minimum size of the grizzly bear population reached around that time was merely coincidental with conservation

actions, in which case the data should be analyzed all together. Perform this analysis and estimate the ultimate extinction probability, the probability of extinction in 100 years, and the mean time to extinction, starting with $n_0 = 99$ corresponding to 1997 (the last year in Table 1 of the chapter).

Following the procedure on page 15, we first transform counts n to x by taking natural logarithms, yielding the following transformed data:

Year (t)	x_t	Year (t)	x_t	Year (t)	x_t
1959	3.78419	1974	3.583519	1989	4.174387
1960	3.850148	1975	3.526361	1990	4.304065
1961	3.828641	1976	3.663562	1991	4.234107
1962	3.78419	1977	3.555348	1992	4.174387
1963	3.828641	1978	3.526361	1993	4.043051
1964	3.806662	1979	3.637586	1994	4.248495
1965	3.828641	1980	3.583519	1995	4.394449
1966	3.688879	1981	3.610918	1996	4.59512
1967	3.663562	1982	3.713572	1997	4.59512
1968	3.663562	1983	3.663562		
1969	3.73767	1984	3.931826		
1970	3.663562	1985	3.850148		
1971	3.713572	1986	4.043051		
1972	3.688879	1987	3.871201		
1973	3.496508	1988	4.094345		

Table 1: Transformed estimates ($x_t = \ln(n_t)$) of the number of adult female grizzly bears in the Greater Yellowstone Ecosystem, 1959-1997.

Next, we compute growth/decline increments $y_i = x_{i+1} - x_i$. Note that we "lose" a data point here because there is no x_{1998}, which would be required to calculate y_{1997}.

Because these data are regularly sampled (each time interval is the same – 1 year), this exercise can be completed by hand. The mean of y is

$$\bar{y} = \frac{1}{38} \sum_{1959}^{1996} y_i = 0.02134027. \tag{9}$$

From the reading, we know $\bar{y} = r - v_2/2$ giving $r - v_2/2 = 0.02134027$, which we can rearrange to produce an *estimator* of r:

$$\hat{r} = 0.02134027 + v_2/2. \tag{10}$$

The sample variance of y is an estimate of v_2.

$$\hat{v}_2 = \sigma^2 = \frac{1}{38} \sum_{1959}^{1996} (y_i - \bar{y})^2 = 0.01320551. \tag{11}$$

Year (t)	y_t	Year (t)	y_t	Year (t)	y_t
1959	0.06595797	1974	-0.05715841	1989	0.14310084
1960	-0.02150621	1975	0.13720112	1990	-0.08338161
1961	-0.04445176	1976	-0.10821358	1991	-0.05971923
1962	0.04445176	1977	-0.02898754	1992	-0.13133600
1963	-0.02197891	1978	0.11122564	1993	0.20544397
1964	0.02197891	1979	-0.05406722	1994	0.14595391
1965	-0.13976194	1980	0.02739897	1995	0.20067070
1966	-0.02531781	1981	0.10265415	1996	0.00000000
1967	0	1982	-0.05001042		
1968	0.07410797	1983	0.26826399		
1969	-0.07410797	1984	-0.08167803		
1970	-0.02469261	1985	0.19290367		
1971	0.05001042	1986	-0.17185026		
1972	-0.19237189	1987	0.22314355		
1973	0.08701138	1988	0.08004271		

Table 2: Growth/decline increments of the number of adult female grizzly bears in the Greater Yellowstone Ecosystem, 1959-1997.

Substituting our estimated value of v_2 in equation 10 yields $\hat{r} = 0.02134027 + 0.01320551/2 = 0.02794302$.

Starting with $n_0 = n_{1997} = 99$ and substituting into equation 22 from the chapter gives the ultimate probability of extinction:

$$\begin{aligned} p_0(\infty) &= \exp(-2 \ln n_0 (r - v_2/2)/v_2) \\ &= \exp(-2 \ln 99 (0.02794302 - 0.01320551/2)/0.01320551) \\ &\approx 3.5 \times 10^{-7}. \end{aligned}$$

The probability of extinction at time $t = 100$ years, given that it occurs at all, can be calculated from equation 22:

$$\begin{aligned} p_1(t = 100) &= \frac{\ln n_0}{\sqrt{2\pi v_2 t^3}} \exp\left(-\frac{(\ln n_0 - |r - v_2/2|t)^2}{2 v_2 t} \right) \\ &= \frac{\ln 99}{\sqrt{2\pi (0.01320551) 100^3}} \exp\left(-\frac{(\ln 99 - |(0.02794302) - 0.01320551/2|100)^2}{2(0.01320551)(100)} \right) \\ &= 0.001610047. \end{aligned}$$

If we want the cumulative conditional extinction probability (i.e., the probability of extinction at $t = 1$ or $t = 2$ or $t = 3$..., let's call it P_{100}) we need to sum this up for all values of t between 1 and 100. (This is an approximation that uses the midpoint rule in calculus.)

$$P_{100} = \sum_{i=1}^{100} p(i) \approx 0.0236$$

To get the unconditional probability of extinction in 100 years, we mutiply by the ultimate chance of extinction, yielding

$$\text{Cumulative probability of extinction in 100 years} \approx 0.0236 \times 3.5 \times 10^{-7} \approx 8.3 \times 10^{-9}$$

Finally, we calculate the mean time to extinction as in homework problem 2.

$$\begin{aligned} m(t) &= \ln(n_0)/|r - v_2/2| \\ &= \ln 99/(0.02794302 - 0.01320551/2) \\ &\approx 215 \text{ years} \end{aligned}$$

4. Devise a test to determine if the Endangered Species Act and other measures taken around 1973 had a statistically detectable effect on the population dynamics of grizzly bears. Test for effects on both mean and variation in change in population size over time.

   The change in population size over time is indicated by the increments y_i. We can test for a difference in the means using a t-test. In R,

```
> t.test(y[1:15], y[16:38])

        Welch Two Sample t-test

data:  y[1:15] and y[16:38]
t = -1.6945, df = 35.846, p-value = 0.09884
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -0.12602277  0.01130138
sample estimates:
  mean of x   mean of y
-0.01337805  0.04398265
```

   This test fails (at the $\alpha = 0.05$ level) to reject the null hypothesis that there was no difference in the average growth rate of the population before vs. after 1973.

   An F-test can be used to compare the variances of the increments y. In R,

```
> var.test(y[1:15], y[16:38])

        F test to compare two variances

data:  y[1:15] and y[16:38]
F = 0.36202, num df = 14, denom df = 22, p-value = 0.05403
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.1431775 1.0187074
sample estimates:
ratio of variances
         0.3620216
```

   This test also fails (at the $\alpha = 0.05$ level) to reject the null hypothesis that there was no difference before vs. after 1973.